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Applications of Thirring model to inhomogenous rolling tachyon and dissipative quantum mechanics

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Abstract: We study the rolling tachyon and the dissipative quantum mechanics using the Thirring model with a boundary mass. We construct a boundary state for the dissipative quantum system in one dimension, which describes the system at the off-critical points as well as at the critical point. Then we extend the Thirring model with a boundary mass in order to depict the time evolution of an unstable D-branes with one direction wrapped on a circle of radius R, which is termed the inhomogeneous rolling tachyon. The analysis based on the Thirring model shows that the time dependent evolution of the inhomogeneous tachyon is possible only when $\frac{2}{\sqrt{3}} < R < 2$.

Keywords: Field Theories in Lower Dimensions, Tachyon Condensation.

Contents

1. Introduction

The two space-time dimensional quantum field theory of massless bosons with a periodic boundary potential has recently come into the spotlight again. This boundary conformal field theory has received constant attention along the years, since it describes the various condensed physics systems such as the dissipative quantum mechanics of a particle in a one-dimensional periodic potential [1 – 5], Josephson junction arrays [6 – 8] and the dissipative Hofstadter problem [9, 10]. The applications of the theory also include the Kondo problem [11, 12], the one-dimensional conductors [13], tunneling between Hall edge states [14], and junctions of quantum wires [15]. The recent revival of interest in the theory is mainly due to the string theory applications. As the string with its ends on an unstable D-brane develops a marginal boundary interaction, the tachyon field of the open string condensates [16 – 21]. This process, called "rolling tachyon" [22 – 53] is believed to be responsible for the decay of the unstable D-brane.

Recently, the theory with a periodic boundary potential is discussed in detail by applying the fermionization technique [54, 55]. The advantage of the fermion formulation is that one can explicitly construct the boundary state for the rolling tachyon, since the periodic boundary potential becomes a boundary fermion mass term, which is quadratic in the fermion field. The fermion formulation of the theory, however, has been performed only for the theory at the critical point. In this paper we apply the fermion formulation to the theory at off-critical points to develop a perturbation theory around the critical points. In the fermion formulation, the theory at off-critical points is described by the Thirring model with a boundary mass. The fermion formulation of the theory at off-critical points would also be useful in discussing the inhomogeneous rolling tachyon.

The massless Thirring model [56], which is the first example of an exactly solvable relativistic interacting field theory, has served as an excellent laboratory for the study of various aspects of the quantum field theory in two dimensions. Since the seminal work of

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Thirring, extensive studies of the model have been carried out by numerous authors [57–60]; notably the complete solution of the theory was obtained by Klaiber [60]. Along this line, Schwinger [61] obtained an exact solution of quantum electrodynamics in 1+1 dimensions and Coleman [62] proved the equivalence between the sine-Gordon model and the massive Thirring model.

Here we widen the range of applications of the Thirring model by studying the dissipative quantum system and the inhomogeneous rolling tachyon in terms of it. The Thirring model with a boundary mass is found to be the most suitable framework to discuss the dissipative quantum system: At the critical point the Thirring coupling, which is directly related to the friction constant of the system, vanishes and the theory reduces to a free fermion theory with a boundary mass. The boundary state takes a simple form at the critical point when it is expressed in fermion variables as discussed in refs. [54, 55, 63]. Thus, the Thirring model provides a perturbation theory expanded in the Thirring coupling near the critical point. One of advantages of the boundary state formulation is that all perturbative corrections can be easily shown to vanish at the critical point.

Recently, Sen [25] discussed a Dp-brane of the bosonic string theory with one direction wrapped on a cricle of radius $R > 1$. Since the boundary interaction depends on the additional spatial compact coordinate Y, we expect that the tachyon condensation may be inhomogeneous. In this paper we show that the inhomogeneous rolling tachyon can be also studied within the framework of the Thirrring model and the boundary state formulation.

2. The Schmid model and the Thirring model

Caldeira and Leggett [64, 65] discussed first the quantum mechanical description of dissipation by introducing a bath or environment, which consists of an infinite number of harmonic oscillators, coupled to the system. In the quantum theory, the interaction with the bath produces a non-local effective interaction. Subsequently, Schmid [1] studied the dissipative system in the presence of a periodic potential. The one dimensional dissipative model with a periodic potential, called the Schmid model, is described by the following action

$$S_{SM} = \frac{\eta}{4\pi\hbar}\int_{-T/2}^{T/2} dt dt' \frac{(X(t)-X(t'))^2}{(t-t')^2} - \frac{V_0}{\hbar}\int_{-T/2}^{T/2} dt \cos\frac{2\pi X}{a}. \tag{2.1}$$

The first non-local term is responsible for the dissipation, and the second term denotes the periodic potential respectively. An interesting feature of the model is that it exhibits a phase transition, which is unlike one dimensional quantum mechanical systems with local interactions only. Depending on the value of the friction constant η, the phase diagram of the system divides into two phases; the localized phase and the delocalized one.

We can map the Schmid model to the string theory on a disk by identifying the time as the boundary parameter σ in string theory and scaling the field variable X as follows:

$$t = \frac{T}{2\pi}\tau, \quad X \to \frac{a}{2\pi}X. \tag{2.2}$$

Then, the action for the Schmid model reads as

$$S_{SM} = \frac{\eta}{4\pi\hbar}\left(\frac{a}{2\pi}\right)^2 \int d\tau d\tau' \, \frac{(X(\tau)-X(\tau'))^2}{(\tau-\tau')^2} - \frac{V_0}{\hbar}\frac{T}{2\pi}\int d\tau \, \frac{1}{2}\left(e^{iX}+e^{-iX}\right). \quad (2.3)$$

This action can be interpreted as the boundary effective action for the open bosonic string subject to a boundary periodic potential on a disk with a boundary condition; on the boundary ∂M, $X(\sigma,\tau) = X(\tau)$,

$$e^{-iS_{SM}} = \int D[X] \exp\left[-i\left(\frac{1}{4\pi\alpha'}\int_M d\tau d\sigma \partial_\alpha X \partial^\alpha X - \frac{m}{2}\int_{\partial M} d\tau \left(e^{iX}+e^{-iX}\right)\right)\right]. \quad (2.4)$$

Here, we identify the physical parameters of the two theories as

$$\frac{\eta}{4\pi\hbar}\left(\frac{a}{2\pi}\right)^2 = \frac{1}{8\pi^2\alpha'}, \quad -\frac{V_0}{\hbar}\frac{T}{2\pi} = m. \quad (2.5)$$

In the string theory, the periodic potential describes the interaction between the open string and the unstable D-brane.

The open string dynamics is often described more efficiently in its equivalent closed string picture by the boundary state formulation. The corresponding closed string action is obtained from its open string action by simply taking $\sigma \to \tau$, $\tau \to \sigma$,

$$S = \frac{1}{4\pi\alpha'}\int d\tau d\sigma \, \partial_\alpha X \partial^\alpha X - \frac{m}{2}\int d\sigma \left(e^{iX}+e^{-iX}\right). \quad (2.6)$$

The open string dynamics can be encoded completely by the boundary state.

When $\alpha = 1/\alpha' = 1$, the system becomes critical. This can be easily understood if we introduce an auxiliary boson field Y and fermionize the system [54, 55, 66]. Introducing an auxiliary boson field Y which satisfies the Dirichlet condition $Y|_{\partial M} = 0$ at the boundary, and defining the boson fields, $\phi_1 = \frac{X+Y}{\sqrt{2}}$, $\phi_2 = \frac{X-Y}{\sqrt{2}}$, we may rewrite the action as

$$S = \frac{\alpha}{4\pi}\int_M d\tau d\sigma \sum_i^2 \partial\phi_i \partial\phi_i - \frac{m}{4}\int_{\partial M} d\tau \sum_i^2 \left(e^{i\sqrt{2}\phi_i} + e^{-i\sqrt{2}\phi_i}\right). \quad (2.7)$$

Since Y is a free boson field and it vanishes at the boundary, Y is completely decoupled from the physical degrees of freedom. We see that if $\alpha = 1$, $e^{\pm i\sqrt{2}\phi_i}$ are marginal boundary operators with the scaling dimension 1. An explicit calculation of the current correlation function or the mobility shows that the theory becomes indeed critical where $\alpha = 1$

$$\langle 0|\partial_\sigma X(\sigma)\partial_\sigma X(\sigma')|B\rangle = -\frac{1}{2}(1-\pi^2 m^2)\sin^{-2}\frac{(\sigma-\sigma')}{2}. \quad (2.8)$$

The boundary state at the critical point can be explicitly evaluated if the model is fermionized; the boson fields are mapped to the fermion fields as

$$\psi_{1L}(z) = \zeta_{1L} : e^{-\sqrt{2}i\phi_{1L}(z)} :, \quad \psi_{2L}(z) = \zeta_{2L} : e^{\sqrt{2}i\phi_{2L}(z)} : \quad (2.9a)$$
$$\psi_{1R}(\bar z) = \zeta_{1R} : e^{\sqrt{2}i\phi_{1R}(\bar z)} :, \quad \psi_{2R}(\bar z) = \zeta_{2R} : e^{-\sqrt{2}i\phi_{2R}(\bar z)} : \quad (2.9b)$$

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where $\zeta_{iL/R}$ are co-cycles, ensuring the anti-commutation relations between the fermion operators. Since the boundary interaction term can be written as a boundary fermion mass term, which is only quadratic in fermion field, the model is exactly solvable

$$S = \int \frac{d\tau d\sigma}{2\pi} \left(\bar{\psi}_1 \gamma^\mu \partial_\mu \psi_1 + \bar{\psi}_2 \gamma^\mu \partial_\mu \psi_2\right) + m \int \frac{d\sigma}{2\pi} \left(\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2\right) \tag{2.10}$$

where $\psi_i = (\psi_{iL}, \psi_{iR})^t$, and

$$\gamma^0 = \sigma_1, \quad \gamma^1 = \sigma_2, \quad \gamma^5 = \sigma_3 = -i\gamma^0\gamma^1. \tag{2.11}$$

The boundary state is given formally as

$$|B\rangle =: \exp\left[m \int \frac{d\sigma}{2\pi} \left(\bar{\psi}_1 \psi_1 + \bar{\psi}_2 \psi_2\right)\right] : |N, D\rangle \tag{2.12}$$

where $|N, D\rangle$ is a simple boundary state satisfying

$$\left(\psi_R(0,\sigma) + i\sigma^2 \psi_L(0,\sigma)\right) |N, D\rangle = 0, \quad \left(\psi_R^\dagger(0,\sigma) + i\psi_L^\dagger(0,\sigma)\sigma^2\right) |N, D\rangle = 0. \tag{2.13}$$

We refer the reader to ref. [55] for the explicit expression of the boundary state $|B\rangle$.

Now let us discuss the dissipative system off the critical points. When $\alpha \neq 1$, we may write the action as

$$\begin{aligned} S = &\frac{1}{4\pi} \int_M d\tau d\sigma \sum_i^2 \partial\phi_i \partial\phi_i + \frac{1}{4\pi}(\alpha - 1) \int_M d\tau d\sigma \sum_i^2 \partial\phi_i \partial\phi_i \\ &- \frac{m}{4} \int_{\partial M} d\tau \sum_i^2 \left(e^{i\sqrt{2}\phi_i} + e^{-i\sqrt{2}\phi_i}\right). \end{aligned} \tag{2.14}$$

and treat the second term as an interaction. In terms of the fermion fields the second term can be written as the Thirring interaction term. Hence, the fermionized action is given by

$$S = \frac{1}{2\pi} \int_M d\tau d\sigma \sum_i^2 \left(\bar{\psi}_i \gamma^\mu \partial_\mu \psi_i + \frac{g}{4\pi} j_i^\mu j_{i\mu}\right) + \frac{m}{2} \int_{\partial M} d\sigma \sum_i^2 \bar{\psi}_i \psi_i \tag{2.15}$$

where $g = \pi(\alpha - 1)$. This is the Thirring model with a boundary mass. At the critical point where $g = 0$ ($\alpha = 1$), the action reduces to the free fermion theory with a boundary mass. Near the critical point, we can use this Thirring action to develop a perturbation theory for the dissipative quantum system. We note that the theory at the critical point corresponds to the homogeneous rolling tachyon of the string theory [22].

3. Boundary state near the critical point

In order to apply the boundary state formulation to the dissipative system near the critical point, the bulk action should be free. We may transmute the bulk Thirring interaction into a boundary one by introducing Abelian gauge fields

$$S = \frac{1}{2\pi} \int_M d\tau d\sigma \sum_i^2 \left[\bar{\psi}_i \gamma^\mu \left(\partial_\mu + iA_{i\mu}\right) \psi_i + \frac{\pi}{g} A_{i\mu} A_i{}^\mu\right] + \frac{m}{2} \int_{\partial M} d\sigma \sum_i^2 \bar{\psi}_i \psi_i. \tag{3.1}$$

In general, the Abelian gauge vector fields in $1+1$ dimensions may be decomposed as

$$A_i^\mu = \epsilon^{\mu\nu}\partial_\nu\theta_i + \partial^\mu\chi_i, \quad i=1,2. \tag{3.2}$$

Hence, the interaction between the gauge fields and the fermion fields may be removed by a gauge transformation

$$\psi_i = e^{-\gamma_5\theta_i - i\chi_i}\psi_{i\,0}, \quad \bar{\psi}_i = \bar{\psi}_{i\,0}e^{-\gamma_5\theta_i + i\chi_i}. \tag{3.3}$$

Then the bulk action becomes a free field one

$$S_{bulk} = \frac{1}{2\pi}\int_M d\tau d\sigma \sum_{i=1}^{2}\left[\bar{\psi}_{i\,0}\gamma^\mu\partial_\mu\psi_{i\,0} + \left(\frac{\pi}{g}+1\right)(\partial\theta_i)^2 + \frac{\pi}{g}(\partial\chi_i)^2\right]. \tag{3.4}$$

The additional kinetic action for θ_i is a manifestation of the $\mathrm{U}(1)$ chiral anomaly:

$$D[\psi]D[\bar{\psi}] = D[\psi_0]D[\bar{\psi}_0]\exp\left[\frac{1}{4\pi}\int d\tau d\sigma\sum_i(\partial\theta_i)^2\right]. \tag{3.5}$$

Since the boundary mass term is not invariant under the $\mathrm{U}(1)$ chiral gauge transformation, it transforms as

$$\sum_i\bar{\psi}_i\psi_i = \sum_i\bar{\psi}_{i0}e^{-2\gamma_5\theta_i}\psi_{i0}. \tag{3.6}$$

Note that scalar fields χ_i are free in the bulk and do not appear in the boundary action. Since the physical operators, being $\mathrm{U}(1)_V$ gauge invariant, do not depend on χ_i, we may drop them. For the sake of convenience, we scale the scalar fields

$$\theta_i \to \kappa\theta_i, \quad \kappa = \sqrt{\frac{g}{2(\pi+g)}}. \tag{3.7}$$

It brings us to

$$S = \frac{1}{2\pi}\int_M d\tau d\sigma\sum_{i=1}^{2}\left[\bar{\psi}_{i\,0}\gamma^\mu\partial_\mu\psi_{i\,0} + \frac{1}{2}(\partial\theta_i)^2\right] + m\int_{\partial M} d\sigma\sum_i\bar{\psi}_{i0}e^{-2\gamma_5\kappa\theta_i}\psi_{i0}. \tag{3.8}$$

It is clear that in the limit of the critical point, $\kappa \to 0$, the interaction between the scalar fields θ_i and ψ_0 vanishes. Thus, θ_i, becoming free fields, can be dropped and the action reduces to that for free fermions with boundary masses at the critical point.

The next step to construct the perturbative boundary state formulation is to find appropriate boundary conditions for ψ_0^i and θ_i. We note that the boundary conditions for the fermion fields ψ_0^i should coincide with those at the critical point. So the boundary conditions for them are the same as those in eq. (2.13). The boundary conditions for the fermion fields would remain intact as the Thirring interaction term is turned on, since the boundary conditions for the boson fields ϕ_i do not change. The boundary conditions for the fermion fields can be formally kept unchanged if we require the boson fields to satisfy the following conditions

$$\theta_1|B_0\rangle = -\theta_2|B_0\rangle, \quad \chi_1|B_0\rangle = \chi_2|B_0\rangle. \tag{3.9}$$

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Since χ_i are completely decoupled from the physical degrees of freedom, the boundary conditions for χ_i are not important. Note that eq. (3.9) only fixes the boundary condition for $\frac{1}{\sqrt{2}}(\theta_1 + \theta_2)$. We may choose Neumann conditions for $\frac{1}{\sqrt{2}}(\theta_1 - \theta_2)$ for the sake of completeness. Once, the simple boundary state $|B_0\rangle$ is constructed, the boundary state for the dissipated system $|B(m,\kappa)\rangle$ may be given as

$$|B(m,\kappa)\rangle = \exp\left[m \int_{\partial M} d\sigma \sum_i \bar{\psi}_i e^{-2\gamma_5 \kappa \theta_i} \psi_i\right] |B_0\rangle \tag{3.10}$$

where we drop the subscript "$_0$" of the fermion fields for notational convenience.

It is interesting to see that the scalar fields θ_i appear only through the boundary interaction and the physical operators such as currents do not depend upon them. Near the critical point where $|\kappa| < 1$, we may expand the boundary state $|B(m,\kappa)\rangle$ in κ as follows

$$\begin{aligned} |B(m,\kappa)\rangle &= \exp\left[m \int d\sigma \sum_i \bar{\psi}_i \psi_i - 2m\kappa \int d\sigma \sum_i \bar{\psi}_i \gamma^5 \psi_i \theta_i\right] |B(0,0)\rangle \\ &= \sum_n \frac{(-2m\kappa)^n}{n!} \left[\int d\sigma \sum_i \bar{\psi}_i \gamma^5 \psi_i \theta_i\right]^n |B(m,0)\rangle \end{aligned} \tag{3.11}$$

where $|B(m,0)\rangle$ corresponds to the boundary state at the critical point, of which explicit expression can be found in ref. [55].

As we expand $|B(m,\kappa)\rangle$ in κ, we encounter a divergent term, proportional to the boundary mass term with a coefficient

$$\begin{aligned} &2m^2\kappa^2 \int d\sigma_1 \int d\sigma_2 \sum_i \langle \theta_i(\sigma_1)\theta_i(\sigma_2)\rangle \Big(\langle \bar{\psi}\gamma^5\psi(\sigma_1)\bar{\psi}\gamma^5\psi(\sigma_2)\rangle \\ &\qquad = 2m^2\kappa^2 \int d\sigma_1 \int d\sigma_2 \ln\left|1 - \frac{z_1}{z_2}\right| \left(\frac{1}{z_1 - z_2}\frac{1}{\bar{z}_1 - \bar{z}_2}\right) \end{aligned} \tag{3.12}$$

where $z_i = e^{i\sigma_i}$ and $\bar{z}_i = e^{-i\sigma_i}$. In order to regularize it we may deform the integration contours for z_i, introducing an infinitesimal parameter $|\epsilon| \ll 1$ as follows

$$z_1 = e^{i\sigma_1}, \quad \bar{z}_1 = \frac{1}{z_1}, \quad z_2 = e^{-\epsilon} e^{i\sigma_2}, \quad \bar{z}_2 = \frac{e^{-2\epsilon}}{z_2}. \tag{3.13}$$

Then we find,

$$\int d\sigma_1 \int d\sigma_2 \ln\left|1 - \frac{z_1}{z_2}\right| \left(\frac{1}{z_1 - z_2}\frac{1}{\bar{z}_1 - \bar{z}_2}\right) = \frac{\pi^2}{2\epsilon}. \tag{3.14}$$

This divergence can be remedied by renormalization of the boundary mass. It yields

$$m = m_0 \left[1 + \frac{\alpha - 1}{2\alpha} \ln\frac{\Lambda^2}{\mu^2}\right] = m_0 \left(\frac{\Lambda^2}{\mu^2}\right)^{\frac{(\alpha-1)}{2\alpha}} \tag{3.15}$$

where $\ln\frac{\Lambda^2}{\mu^2} = \frac{m_0^2\pi^2}{\epsilon}$. The result is in complete agreement with the previous work ref. [3]: If $\alpha > 1$, m tends to grow and if $\alpha < 1$, it scales to zero. The correction to m vanishes when $\alpha = 1$. In fact, an explicit construction of the boundary state shows that perturbative corrections to m vanish at all orders if $\alpha = 1$.

4. Inhomogeneous rolling tachyon

In order to show that there exists a one parameter family of inequivalent solutions which describe the rolling of a generic open string tachyon away from its maximum, Sen [25] discussed a D-p-brane of the bosonic string theory with one direction wrapped on a circle of radius $R > 1$. In addition to the usual tachyonic mode of mass$^2 = -1$, this system has a tachyonic mode of mass

$$R^{-2} - 1 = -m^2, \tag{4.1}$$

which comes from the first momentum mode of the standard tachyon along the circle direction. If we denote the Wick rotated time coordinate by X and the coordinate along the circle by Y, the conformal field theory associated with the rolling tachyon is obtained by perturbing the free conformal field theory by the boundary operator

$$g \int d\sigma \cos(mX) \cos\left(\frac{Y}{R}\right). \tag{4.2}$$

Thus, the rolling tachyon for the system with a compact spatial coordinate is described by the following action

$$S = \frac{1}{4\pi\alpha'} \int_M d\tau d\sigma \left(\partial X \partial X + \partial Y \partial Y\right) + g \int_{\partial M} d\sigma \cos(mX) \cos\left(\frac{Y}{R}\right). \tag{4.3}$$

Since the boundary perturbation depends not only on X but also on the spatial coordinate Y, we expect that the tachyon condensation may be inhomogeneous.

It may be convenient to rewrite the action in terms of scalar fields, ϕ_i, $i = 1, 2$ defined as

$$\begin{aligned} \phi_1 &= mX + \frac{Y}{R} = \cos\theta\, X + \sin\theta\, Y, \\ \phi_2 &= mX - \frac{Y}{R} = \cos\theta\, X - \sin\theta\, Y \end{aligned} \tag{4.4}$$

where

$$m = \cos\theta, \quad \frac{1}{R} = \sin\theta. \tag{4.5}$$

Rewriting the action in terms of ϕ_i, we have

$$S = \frac{1}{4\pi} \int_M d\tau d\sigma \left[\partial\phi_a \partial\phi_a + g^{ab} \partial\phi_a \partial\phi_b\right] + \frac{g}{4} \int_{\partial M} d\sigma \sum_a \left(e^{i\phi_a} + e^{-i\phi_a}\right) \tag{4.6}$$

where

$$g^{11} = g^{22} = \left(\frac{1}{\alpha' \sin^2(2\theta)} - 1\right), \quad g^{12} = g^{21} = -\frac{1}{\alpha'} \frac{\cos 2\theta}{\sin^2(2\theta)}. \tag{4.7}$$

It is easy to see that when $g^{ab} = 0$, i.e.,

$$\alpha' = 1, \quad \theta = \frac{\pi}{4}, \tag{4.8}$$

the action reduces to a critical theory with a sum of two commuting exactly marginal perturbations

$$S = \frac{1}{4\pi}\int_M d\tau d\sigma\, \partial\phi_a \partial\phi_a + \frac{g}{4}\int_{\partial M} d\sigma \sum_a \left(e^{i\phi_a} + e^{-i\phi_a}\right). \tag{4.9}$$

The critical theory becomes a two flavor SU(2)×SU(2) free fermion theory with a boundary mass if fermionized. The time evolution of the rolling tachyon at this critical point has been discussed by Sen [25].

As in the homogeneous tachyon condensation, it would be convenient to employ the fermionization technique to analyze the critical behavior of the system. It begins with diagonalizing the bulk action. The bulk action can be diagonalized by a similarity transformation for the fields ϕ_a

$$\phi_a = \sum_b M_{ab}\phi'_b, \quad M = M^\dagger = M^{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \qquad M^t M = I. \tag{4.10}$$

By the similarity transformation the Lagrangian is diagonalized as

$$L_0 = \frac{1}{4\pi}\sum_a \partial\phi'_a \partial\phi'_a + \frac{1}{4\pi}\sum_a \lambda_a \partial\phi'_a \partial\phi'_a \tag{4.11}$$

where

$$\lambda_1 = \frac{1}{2\alpha' \cos^2\theta} - 1, \quad \lambda_2 = \frac{1}{2\alpha' \sin^2\theta} - 1. \tag{4.12}$$

Then we may introduce auxiliary boson fields φ'_a, $i =, 1, 2$ for which the action has the same form as that for ϕ'_a

$$L_0 = \frac{1}{4\pi}\sum_a (1+\lambda_a)\,\partial\phi'_a \partial\phi'_a + \frac{1}{4\pi}\sum_a (1+\lambda_a)\,\partial\varphi'_a \partial\varphi'_a. \tag{4.13}$$

Since the bulk action for φ'_a is the free field one, φ'_a would decouple from ϕ'_a if the Dirichlet boundary condition is chosen for φ'_a. It also follows from that φ_a fields would decouple from the physical fields ϕ_a since fields ϕ_a and φ_a are related to fields ϕ'_a and φ'_a respectively by a linear transformation.

Defining Φ'_{ai} as follows

$$\Phi'_{a1} = \frac{1}{\sqrt{2}}\left(\phi'_a + \varphi'_a\right), \quad \Phi'_{a2} = \frac{1}{\sqrt{2}}\left(\phi'_a - \varphi'_a\right), \quad a = 1, 2, \tag{4.14}$$

we may rewrite the bulk Lagrangian as

$$L_0 = \frac{1}{4\pi}\sum_{a,i} \partial\Phi'_{ai}\partial\Phi'_{ai} + \frac{1}{4\pi}\sum_{a,i} \lambda_a \partial\Phi'_{ai}\partial\Phi'_{bi} \tag{4.15}$$

Taking the similarity transformation again

$$\Phi_{ai} = M^{-1}_{ab}\Phi'_{bi}, \quad \varphi_{ai} = M^{-1}_{ab}\varphi'_{bi}, \tag{4.16}$$

we have

$$L_0 = \frac{1}{4\pi}\sum_a\sum_i \partial\Phi_{ai}\partial\Phi_{ai} + \frac{1}{4\pi}\sum_{a,b}\sum_i g^{ab}\partial\Phi_{ai}\partial\Phi_{bi} \quad .$$

Now we are in the position to fermonize the system. Introducing the fermion fields defined as

$$\begin{aligned}\psi_{a1L} &= \zeta_{a1L} : e^{-i\sqrt{2}\Phi_{a1L}} :, \quad \psi_{a2L} = \zeta_{a2L} : e^{i\sqrt{2}\Phi_{a2L}} : \\ \psi_{a1R} &= \zeta_{a1R} : e^{i\sqrt{2}\Phi_{a1R}} :, \quad \psi_{a2R} = \zeta_{a2R} : e^{-i\sqrt{2}\Phi_{a2R}} :,\end{aligned} \tag{4.17}$$

we find that the bulk action can be written as

$$S_M = \frac{1}{2\pi}\int_M d\tau d\sigma \left(\sum_{a,i} \bar{\psi}_{ai}\gamma^\mu\partial_\mu\psi_{ai} + \sum_{a,b}\sum_i \frac{g^{ab}}{4} j^\mu{}_{ai} j_{\mu bi} \right) . \tag{4.18}$$

Since we choose the Neumann condition as the boundary condition for ϕ_a and the Dirichlet condition for φ_a, the boundary conditions for the fields Φ_{ai} are given as

$$\Phi_{a1L}(0,\sigma)|B_0\rangle = \Phi_{a2R}(0,\sigma)|B_0\rangle, \quad \Phi_{a2L}(0,\sigma)|B_0\rangle = \Phi_{a1R}(0,\sigma)|B_0\rangle. \tag{4.19}$$

It follows that with the chosen boundary condition the boundary potential may be written as

$$L_B = g\sum_a \left(e^{i\phi_a} + e^{-i\phi_a} \right) = g\sum_a \left(e^{i\sqrt{2}\Phi_a} + e^{-i\sqrt{2}\Phi_a} \right). \tag{4.20}$$

This boundary potential term is equivalent to a boundary mass term in the fermion theory. Hence, the fermionization finally brings us to the following generalized Thirring model with a boundary mass

$$S = \frac{1}{2\pi}\int_M d\tau d\sigma \left(\sum_{a,i} \bar{\psi}_{ai}\gamma^\mu\partial_\mu\psi_{ai} + \sum_{a,b}\sum_i \frac{g^{ab}}{4} j^\mu{}_{ai} j_{\mu bi} \right) + \frac{g}{2}\int_{\partial M} d\sigma \sum_{a,i} \bar{\psi}_{ai}\psi_{ai} \ . \tag{4.21}$$

Applying the renormalization group (RG) analysis, developed in the section 3, to this generalized Thirring model, we find that the RG flow of g does not depend on R

$$g = g_0 \left[1 + \frac{1-\alpha'}{2} \ln\frac{\Lambda^2}{\mu^2} \right] = g_0 \left[\frac{\Lambda^2}{\mu^2} \right]^{\frac{1-\alpha'}{2}} . \tag{4.22}$$

The inhomogeneous rolling tachyon corresponds to the case where $\alpha' = 1$. However, a more detailed RG analysis reveals that this is not the end of the story: The generalized Thirring model and consequently, the inhomogeneous rolling tachyon, turns out to have non-trivial phase diagrams. The action eq. (4.3) or its A one eq. (4.21) is in fact equivalent to the spin-dependent Tomonaga-Luttinger model with a scattering potential at the origin, which describes quantum transport through a single barrier in a one-dimensional interacting electron system as discussed by Furusaki and Nagaosa [4]. When we deal with

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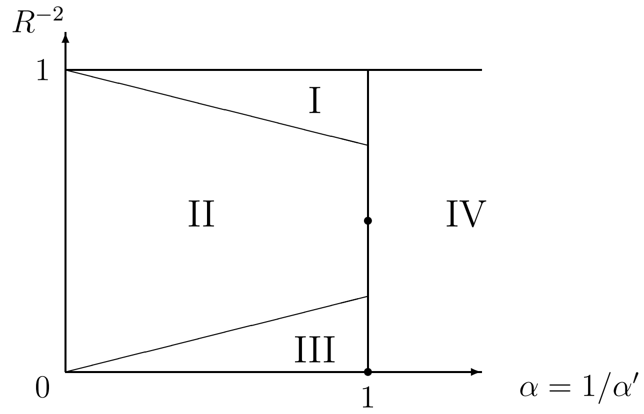

Figure 1: Phase Diagram for The Generalized Thirring Model

the perturbation theory, we must also consider the renormalization of the descendents of the boundary mass operators such as $g_+ e^{i\sqrt{2}(\Phi_1+\Phi_2)}$ and $g_- e^{i\sqrt{2}(\Phi_1-\Phi_2)}$. Applying the RG analysis to these descendent operators, which arise in the second-order perturbation, we obtain the RG equations for $g_\pm$,

$$g_+ = g_{+0} \left[\frac{\Lambda^2}{\mu^2}\right]^{1-4\alpha'\left(1-R^{-2}\right)}, \quad g_- = g_{-0} \left[\frac{\Lambda^2}{\mu^2}\right]^{1-4\alpha' R^{-2}}. \tag{4.23}$$

Thus, the RG equations for the boundary mass operator and its descendents yield that the phase diagram divides into four regions as shown in figure 1 and the phase boundaries are set by

$$\alpha = \frac{1}{\alpha'} = 1, \quad \frac{1}{R^2} = \frac{\alpha}{4}, \quad \frac{1}{R^2} = 1 - \frac{\alpha}{4}. \tag{4.24}$$

(One may obtain the same phase diagram by transcribing the result of Furusaki and Nagaosa [4] into the string theory.)

In the regions I, II, III and IV the RG flow leads the system to fixed points, which correspond to the boundary states $|D,N\rangle$, $|N,N\rangle$, $|N,D\rangle$ and $|D,D\rangle$ respectively. Here N and D denote Neumann and Dirichlet boundary conditions and the first and second label is the condition for the X and Y bosons, respectively. Consequently if we begin with a Dp-brane with the boundary interaction, due to the tachyon condensation, the RG flow drives the system into Sp-brane, Dp-brane, $D(p-1)$-brane, and $S(p-1)$-brane in the regions I, II, III, and IV respectively.

If we wish to study the inhomogeneous rolling tachyon, we should pay attention to the line, $\alpha = 1$ on the phase diagram figure 1. From the phase diagram figure 1, it is clear that the inhomogeneous rolling tachyon model has three different phases as depicted in figure 2: A) $1 < R < \frac{2}{\sqrt{3}}$, B) $\frac{2}{\sqrt{3}} < R < 2$, C) $R > 2$. The point where $R = \sqrt{2}$ corresponds to the critical point where the model reduces to the $\mathrm{SU}(2) \times \mathrm{SU}(2)$ free fermion theory with boundary masses discussed by Sen [25].

If we take the RG effects into account, we may have to choose a different action for each region to describe the inhomogeneous rolling tachyon in a way that is consistent with

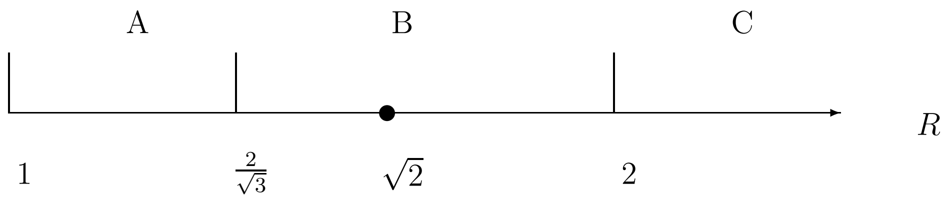

Figure 2: Phase Diagram for Inhomogeneous Rolling Tachyon

the perturbation theory. In the region A, the boundary operator $e^{i\sqrt{2}(\Phi_1+\Phi_2)}$ is a relevant operator while $e^{i\sqrt{2}(\Phi_1-\Phi_2)}$ is an irrelevant one. The RG effects drive the boundary condition for X to be Dirichlet. If the Dirichlet condition is once chosen for X, the action is reduced as

$$S = \frac{1}{4\pi}\int_M d\tau d\sigma\,(\partial X\partial X + \partial Y\partial Y) + \frac{g}{2}\int_{\partial M} d\sigma\left(e^{i\sqrt{1-R^{-2}}X} + e^{-i\sqrt{1-R^{-2}}X}\right). \quad (4.25)$$

Note that the boundary mass operator becomes irrelevant in the region A. Hence, they can be ignored. It follows that the action for the rolling tachyon, compatible with the perturbation theory, is given as a sum of two free boson actions (or a free fermion action equivalently) in the region A

$$S = \frac{1}{4\pi}\int_M d\tau d\sigma\,(\partial X\partial X + \partial Y\partial Y) \quad (4.26)$$

where the boundary conditions for X and Y are Dirichlet and Neumann respectively. Since the boundary conditon for X is chosen to be Dirichlet, the tachyon is not rolling and the theory describes a static Sp-brane. In the region B, both boundary operators $e^{i\sqrt{2}(\Phi_1+\Phi_2)}$ and $e^{i\sqrt{2}(\Phi_1-\Phi_2)}$ are irrelvant, but the boundary mass operators $\bar{\psi}_{ai}\psi_{ai}$ are marginal. Thus, the inhomogeneous tachyon rolls and the rolling tachyon is described by the Thirring action eq. (4.21) with g^{ab} given as follows

$$g^{11} = g^{22} = \left(\frac{1}{\sin^2(2\theta)} - 1\right), \quad g^{12} = g^{21} = -\frac{\cos 2\theta}{\sin^2(2\theta)} \quad (4.27)$$

in the region B. In the region C, the boundary operator $e^{i\sqrt{2}(\Phi_1+\Phi_2)}$ is an irrelevant operator while $e^{i\sqrt{2}(\Phi_1-\Phi_2)}$ is a relevant one. The RG effects drive the boundary condition for Y to be Dirichlet. Then if the Dirichlet condition is chosen as the boundary condition for Y, the action may be written as

$$S = \frac{1}{4\pi}\int_M d\tau d\sigma\,(\partial X\partial X + \partial Y\partial Y) + \frac{g}{2}\int_{\partial M} d\sigma\left(e^{i\frac{Y}{R}} + e^{-i\frac{Y}{R}}\right). \quad (4.28)$$

The boundary operator, $e^{i\frac{Y}{R}} + e^{-i\frac{Y}{R}}$, is irrelevant in the region C. Thus, the action, which is compatible with the perturbation theory, reduces to a sum of two free boson actions eq. (4.26) as in the region A. However, the boundary conditions differ from those in the region A: The boundary conditions for X and Y are Neumann and Dirichlet respectively. The action depicts a static $D(p-1)$-brane instead of a time dependent evolution of the system. Based on the RG analysis, we may conclude that the inhomogeneous tachyon is rolling only when $\frac{2}{\sqrt{3}} < R < 2$. In the region C one may further notice that the boundary mass operators become marginal as $R \to \infty$ (with the Dirichlet boundary condition for Y) and the tachyon condensation may depend on time. This special point corresponds to the homogeneous rolling tachyon discussed in sections 2 and 3.

5. Conclusions

A few remarks are in order to conclude this paper. We first discuss the dissipative quantum system in one dimension and its relation to the homogeneous rolling tachyon, employing the boundary state formulation of string theory and the Thirring model in two dimensions. The framework presented here has some advantages over the previous ones on the dissipative quantum systems. We need to deal with a local boundary interaction only and can take advantage of the string theory techniques to explore various aspects of the system. In particular, the boundary state for the system can be explicitly constructed. Since the system is described in terms of the free fields, all the physical quantities are exactly calculable at any given order.

The boundary state formulation of the Thirring model with a boundary mass is also found to be useful to study the inhomogeneous tachyon [25] and the single-barrier problem in a one-dimensional interacting electron system [4]. We show then if fermionized, both models are described by a generalized Thirring action with boundary masses

$$S = \frac{1}{2\pi}\int_M d\tau d\sigma \left(\sum_{a,i=1}^{2} \bar{\psi}_{ai}\gamma^\mu \partial_\mu \psi_{ai} + \sum_{a,b=1}^{2}\sum_{i=1}^{2} \frac{g^{ab}}{4} j^\mu{}_{ai} j_{\mu bi} \right) + \frac{g}{2}\int_{\partial M} d\sigma \sum_{a,i=1}^{2} \bar{\psi}_{ai}\psi_{ai}. \quad (5.1)$$

The inhomogeneous rolling tachyon is a special case where $\alpha' = 1/\alpha = 1$. From the RG analysis of the boundary mass and its descendent operators in the perturbation theory, we point out that the generalized Thirring model has a non-trivial phase diagram which divides into four region, depending on the string tension and the radius of the compact circle. The RG fixed point for each region corresponds to Sp-brane, Dp-brane, $D(p-1)$-brane, and $S(p-1)$-brane respectively. Concurrently, the inhomogeneous rolling tachyon model also has a non-trivial phase diagram, which divides into three different regions, depending on the radius of the compact circle. Since in the region where $1 < R < \frac{2}{\sqrt{3}}$ and $R > 2$, the boundary mass operators and its descendents have trivial fixed points, corresponding to the Dirichlet state or the Neumann state, the RG flow drives the action to free field one which describes a static Sp-brane or a $D(p-1)$-brane. The inhomogeneous tachyon rolls only when $\frac{2}{\sqrt{3}} < R < 2$.

There are several directions along which this work can be extended. We can construct a fermion Thirring model, which is more suitable to analyze the dynamics of the inhomogeneous rolling tachyon than the bosonic model. The constructed Thirring model would be also useful to study the single-barrier problem in condensed matter physics near the critical points. One of the advantages of the fermion formulation is that it is easier to construct boundary states, which are extremely convenient in discussing various dynamical aspects of the theories. We save more detailed analyses of the inhomogeneous tachyon condensation and the single-barrier problem based on the boundary state formulation for future work [68]. We may also employ the constructed Thirring to evaluate the closed string emission from the decaying Dp-brane through the inhomogeneous tachyon condensation.

The string theory action eq. (2.6) or eq. (2.7) for the dissipative system is also known as the boundary sine-Gordon model, which has been used to study the quantum impurity problems and the edge state tunneling in the fractional quantum Hall effect. See ref. [67] and references there in for discussions in this direction. The quantum integrability plays an important role in the studies along this direction. It may be interesting to explore the relationship between the boundary state formulation given in this paper and the analysis based on the quantum integrability. The quantum integrability of the more general boundary sine-Gordon models, corresponding to the Thirring models for the inhomogeneous tachyon condensation and the single-barrier problem also deserves to be an interesting subject to be studied.

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